PHILOSOPHICAL TRANSACTIONS.

XVI. *Of the Rotatory Motion of a Body of any Form whatever, revolving, without Reſtraint, about any Axis paſſing through its center of Gravity. By Mr.* John Landen, *F.R.S.*

Read March 17, 1785.

A SPHERICAL body, uniformly denſe, it is obvious, will, if made to revolve freely about any axis paſſing through its center, continue to revolve about the ſame axis; and, by what I have ſhewn in the *Philoſophical Tranſactions* for the year 1777, it appears, that a *cylinder* of uniform denſity, whoſe length is to its radius as $\sqrt{3}$ to 1, will do the ſame. It likewiſe appears, by my *Mathematical Memoirs*, that

that a *cone*, a *conoid*, a *prism*, or a *pyramid*, &c. of certain dimensions, will have the like property of continuing, without any restraint, to revolve about any axis passing through its center of gravity.

When the axis, about which a body may be made to revolve, is not a permanent one, the centrifugal force of its particles will disturb its rotatory motion, so as to cause it to change its axis of rotation (and consequently its poles) every instant, and endeavour to revolve about a new one: and I cannot think it will be deemed an uninteresting proposition to determine in what track, and at what rate, the poles of such momentary axis will be varied in any body whatever; as, without the knowledge to be obtained from the solution of such problem, we cannot be certain whether the earth, or any other planet, may not, from the *inertia* of its own particles, so change its momentary axis, that the poles thereof shall approach nearer and nearer to the present equator, or whether the evagation of the momentary poles, arising from that cause, will not be limited by some known lesser circle. Which certainly is an important consideration in astronomy; especially now that branch of science is carried to great perfection, and the acute astronomer endeavours to determine the motions of the heavenly bodies with the greatest exactness possible.

I do not know that the problem has before been solved by any mathematician in these kingdoms; but I am aware that it has been considered by some gentlemen, very eminent for their mathematical knowledge, in other nations. The solutions of it, given by the celebrated M. LEONHARD EULER and M. D'ALEMBERT, I have seen: and we learn from what the last mentioned gentleman has said, in his *Opuscules Mathematiques*,

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that a solution of it, investigated by M. JOHN ALBERT EULER (after a method similar to his father's) obtained the prize given by the Academy of Sciences in the year 1761. The conclusions deduced by those very learned gentlemen differing greatly from mine made me suspect, for some time, that I had somewhere erred in my investigation, and induced me to revise my process again and again with the greatest circumspection. At length my scrutiny has so removed my doubts, that, being well assured of the truth of my theory, I now beg leave to present it to the Royal Society; presuming that it will be found not unworthy of the notice of such readers, as are curious in contemplating the various motions which bodies may naturally have, in consequence of instantaneous or continued impulse.

In the *Philosophical Transactions* referred to above, I gave a specimen of this theory, as far as it relates to the motion of *a spheroid* and *a cylinder*. The improvements I have since made in it, enable me now to extend it to the motion of *any body whatever*, how irregular soever its form may be.

What I here infer therefrom will be found to differ very materially from the deductions in the solutions given by the gentlemen above-mentioned. They represent the angular velocity, and the momentum of rotation of the revolving body, as always *variable*, when the axis about which it has a tendency to revolve is a momentary one, except in a particular case. By my investigation it appears, that the angular velocity and the momentum of rotation will always be *invariable* in any revolving body, though the axis about which it endeavours to revolve be continually varied; and the tracks of the varying poles upon the surface of the body are thereby determined with great facility.

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It is not only obſervable, that the tracks which the varying poles take, in the ſurface of any revolving body, are ſuch that its momentum of rotation may continue the ſame whilſt its angular velocity continues the ſame; but it may be obſerved, that, in any given body, there is only one ſuch track which a momentary pole can purſue from any given point.

If the angular velocity and the momentum of rotation of a revolving body were to vary according to the computations adverted to above, it would follow, that a body might acquire an increaſe of force from its own motion, without being any way affected by any other body whatever, as the ſame percuſſive force, applied at the ſame diſtance from the momentary axis, would not always deſtroy the rotatory motion of the body, which ſurely cannot poſſibly be true. From the principles or laws of motion, which I conſider as undoubtedly true (and which indeed are no other than the common principles of mechanics), I conclude that a revolving body, not affected by any external impulſe, can no more acquire an increaſe in its momentum of rotation, than any other body, moving freely, can acquire an increaſe in its momentum, or quantity of motion, in any given direction, without being impelled by gravity or ſome other force. And the truth of this concluſion (which is hereinafter proved by other reaſoning) may be eaſily inferred from the property of the lever; ſeeing that the joint centrifugal force of the particles of the revolving body (which is the *only* diſturbing force) has no tendency to accelerate or retard their motion about the momentary axis, but only to alter the poſition of ſuch axis, the direction in which that force acts being always in a plane wherein that axis will be found.

By the theory explained in this paper, it appears that a *parallelopipedon* may always be conceived of ſuch dimenſions,

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that being, by ſome force or forces, made to revolve about an axis, paſſing through its center of gravity, with a certain angular velocity, it ſhall move exactly in the ſame manner as any other given body will move, if made to revolve, by the ſame force or forces, about an axis paſſing through its center of gravity; the quantity of matter (as well as the initial angular velocity) being ſuppoſed the ſame in both bodies; and due regard being had, in the application of the moving force or forces, to the correſponding planes in the bodies. Therefore, as we may from thence always aſſign the dimenſions of a *parallelopipedon* that ſhall be affected exactly in the ſame manner as any other given body will be affected, as well with regard to the centrifugal force of the reſpective particles of the bodies, as to the action of equal percuſſive forces, or oſcillation; it will, after ſhewing how the dimenſions of ſuch *parallelopipedon* may be computed, be only neceſſary, in inveſtigating the propoſition under conſideration, to determine the tracks and velocities of the poles of the momentary axis, about which any *parallelopipedon* may be made to revolve.

Firſt then to find ſuch parallelopipedon (P), that, with reſpect to the action of ſuch forces as are mentioned above, it may be affected exactly in the ſame manner as any other given body (Q). Let it be conſidered that G (tab. X. fig. 1.) being the center of gravity, N a point of ſuſpenſion, and O the correſponding center of oſcillation or percuſſion, the rectangle $GN \times GO$ will be an invariable quantity, the direction NGO continuing the ſame; and that a cylindric ſurface being deſcribed, ſuch that the center of the middle circular ſection thereof ſhall be G, and radius $= \sqrt{GN \times GO}$, and whoſe axis ſhall be perpendicular to the plane wherein the line NGO is ſuppoſed to be impelled to move; if all the matter in the body were placed

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any where in that surface, so that G should be the center of gravity of the matter so placed, any given force or forces, acting on the body in the plane just now mentioned, would cause the line NGO in the body to move exactly in the same manner as it would move, if it were carried with the matter placed in the said surface (as before-mentioned) after having been put in motion by the action of the same force or forces. Moreover, let it be considered, that there will at least be three permanent axes of rotation in the body Q, at right angles to each other (as I have proved in my *Mathematical Memoirs*); and that, supposing NGO to coincide with those three axes in three successive cases wherein the matter in Q shall, in each case, be conceived to be placed in a cylindric surface as described above, we may conceive it possible so to place the matter of the body, that all of it shall be in each of those three surfaces, and G still continue its center of gravity. And, a computation being made accordingly, it appears, that the matter of the body Q must be placed, in equal quantities, at each of the eight angular points of a *parallelopipedon* (R) whose dimensions (length, breadth, and thickness) shall be $\sqrt{2d^2 + 2f^2 - 2e^2}$, $\sqrt{2e^2 + 2f^2 - 2d^2}$, and $\sqrt{2d^2 + 2e^2 - 2f^2}$; d, e, and f, being the three values of $\sqrt{GN \times GO}$, when NGO is successively a permanent axis of rotation, with respect to the body Q, in three directions at right angles to each other.

If Q were a parallelopipedon, it may be easily proved, that its dimensions must be $\sqrt{6d^2 + 6f^2 - 6e^2}$, $\sqrt{6e^2 + 6f^2 - 6d^2}$, and $\sqrt{6d^2 + 6e^2 - 6f^2}$, that the corresponding parallelopipedon, at the angular points whereof the matter of Q is conceived to be placed as above, may have the same dimensions as those which we have found our parallelopipedon R must have.

Whence we may infer, that the parallelopipedon (P), which we propofed to find, muft have the dimenfions laft written; namely, length, breadth, and thicknefs, refpectively equal to $\sqrt{6d^2+6f^2-6e^2}$, $\sqrt{6e^2+6f^2-6d^2}$, and $\sqrt{6d^2+6e^2-6f^2}$: which may be confirmed by a more ftrict demonftration founded on the principles made ufe of in my *fourth Memoir*. For it appears by what is there proved, that the centrifugal forces of the particles of any revolving body, in two directions at right angles to each other, may be expreffed in terms of A, B, K, and variable quantities fhewing the pofition of the momentary axis; and that, in a parallelopipedon whofe dimenfions (length, breadth, and thicknefs) are a, b, k; and whofe mafs, or content, is $=$ M; A will be $=\frac{Ma^2}{12}$, B $=\frac{Mb^2}{12}$, and K $=\frac{Mk^2}{12}$. If therefore a be $=\sqrt{6d^2+6f^2-6e^2}$, $b=\sqrt{6e^2+6f^2-6d^2}$, and $k=\sqrt{6d^2+6e^2-6f^2}$; in fuch body,

$$\text{A will be} = \frac{M}{2}\times\overline{d^2+f^2-e^2},$$

$$\text{B} \qquad = \frac{M}{2}\times\overline{e^2+f^2-d^2},$$

$$\text{K} \qquad = \frac{M}{2}\times\overline{d^2+e^2-f^2}.$$

But, in any body whatever,

$$M\times d^2 \text{ is } = \text{ the fum of all the } \overline{x^2+z^2}\times p,$$

$$M\times e^2 \quad = \text{ the fum of all the } \overline{y^2+z^2}\times p,$$

$$M\times f^2 \quad = \text{ the fum of all the } \overline{x^2+y^2}\times p,$$

and $\frac{M}{2}\times\overline{d^2+e^2+f^2}=$ the fum of all the $\overline{x^2+y^2+z^2}\times p$: x, y, and z correfponding to the place of the particle p in the body; x being meafured from the center of gravity upon a permanent axis of rotation, y at right angles to x, and z at right angles

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to y in a plane to which the said axis is perpendicular. Therefore,

A, which is = the sum of all the $x^2 \times p$, will be $= \frac{M}{2} \times \overline{d^2 + f^2 - e^2}$,

B, = the sum of all the $y^2 \times p$, $= \frac{M}{2} \times \overline{e^2 + f^2 - d^2}$,

K, = the sum of all the $z^2 \times p$, $= \frac{M}{2} \times \overline{d^2 + e^2 - f^2}$,

Hence it is evident, that d, e, and f being determined from any body whatever, the values of A, B, and K will be the same in that body as in our parallelopipedon P; and that the centrifugal forces of the particles will be the same in both bodies. Consequently, their motions about successive momentary axes (whose poles are varied by the perturbation arising from those forces), will be the same in both bodies; their initial angular velocities being the same; as well as the position of their initial momentary axes, with respect to the correspondent permanent axes of rotation in each body.

Let us now proceed to find how any *parallelopipedon* will revolve about successive momentary axes passing through its center of gravity: by which means, with the help of the theorem just now investigated, we shall be enabled to define how any body whatever will revolve about such axes; which is the chief purpose of this disquisition.

Fig. 2. and 3. The length, breadth, and thickness of the revolving *parallelopipedon* (P) being $2d$, $2c$, and $2b$, conceive a spherical surface without matter, whose center is the center of gravity of the body P, to be carried about with that body during its motion; and let the said surface be orthographically projected, so that the radius upon which b is measured may be

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represented by AB; the radius upon which d is measured may be represented by AD; and the radius AC, upon which c is measured, may be projected into the central point A. Let P be the momentary pole, and PQ the continuation of the great circle CP. Let a denote the radius AB ($=$AD); g and γ the sine and cosine of the arc CP; s and t the sine and cosine of the arc BQ, to the same radius a; e the angular velocity of the body and spherical surface, measured at the distance a from the momentary axis; and M the mass or content of the parallelopipedon ($=8bcd$).

Then the motive force E, urging the pole P towards Q, will (by what I have proved in my *Mathematical Memoirs*) be $=\frac{Me^2g\gamma}{3a^7}\times\overline{Ds^2-Ca^2}$; and the motive force $\ddot{E}$, urging the same pole in a direction Po, at right angles to that in which E acts, $=\frac{Me^2g}{3a^6}\times Dst$; C and D being equal to c^2-b^2 and d^2-b^2 respectively. Let Pq be to Po as E to $\ddot{E}$; complete the parallelogram oPqr, and draw the diagonal Pr. This last mentioned line will (by what I have shewn in the *Philosophical Transactions* for the year 1777) be perpendicular to the tangent to the polar track at P. Therefore P$p\ddot{p}\dddot{p}$ being the projection of that track, and Pp an indefinitely small particle thereof; if pu be perpendicular to PuA, and the quantities d^2-c^2, c^2-b^2, be not negative; $\frac{\gamma}{a}\times\overline{Ds^2-Ca^2}$ will be to Dst (as Pq to Po) as pu to Pu, the triangles Por and Pup being similar, and $or=$Pq. But with respect to our spherical surface, pu will be to Pu as $\frac{g\dot{s}}{t}$ to $-\frac{a\dot{g}}{\gamma}$; therefore, $\overline{Ca^2-Ds^2}\times\dot{g}$ will be $=Dgs\dot{s}$, and $\frac{\dot{g}}{g}=\frac{Ds\dot{s}}{Ca^2-Ds^2}$. Whence, by taking the

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fluents, we have $s^2 = a^2 \times \frac{Dm^2 - C\gamma^2}{Dg^2}$, and $t^2 = a^2 \times \frac{Dn^2 - B\gamma^2}{Dg^2}$; *m* and *n* denoting the values of *s* and *t*, when *g* is $= a$ and $\gamma = 0$; and B being put to denote the difference $D - C = d^2 - c^2$.

If now β and δ be put to denote the cosines of BP and DP to the radius *a*, we shall, from what is done above, have

$$\beta = \frac{gt}{a} = \frac{\sqrt{Dn^2 - B\gamma^2}}{D^{\frac{1}{2}}},\ \delta = \frac{gs}{a} = \frac{\sqrt{Dm^2 - C\gamma^2}}{D^{\frac{1}{2}}};$$

$$\beta^2 + \gamma^2 + \delta^2 = a^2,\ \beta\dot{\beta} + \gamma\dot{\gamma} + \delta\dot{\delta} = 0;$$

$$b^2\beta^2 + c^2\gamma^2 + d^2\delta^2 = b^2n^2 + d^2m^2, \text{ and } b^2\beta\dot{\beta} + c^2\gamma\dot{\gamma} + d^2\delta\dot{\delta} = 0.$$

Drawing AR so that $D^{\frac{1}{2}} \times$ sine of BR shall be $= C^{\frac{1}{2}}a$, it is *very remarkable*, that the momentary pole (P) will run round about the point B, or about the point D, in the spherical surface, according as the initial pole shall be in the part BCR or DCR of the said surface; that is, according as Dm^2 is less or greater than Ca^2: and that, if the initial pole (P) be any where in the great circle CR, the momentary pole, keeping in the arc of that circle, will continually approach nearer and nearer to the point C in the surface of the sphere; but, by what follows, we shall find that it never can arrive at that point in any finite time!

The equation of the track of the pole in the projection to which we have hitherto referred will, it is now obvious, be $y^2 = \frac{B}{C} \times \overline{x^2 + \frac{Ca^2 - Dm^2}{B}}$; *x*, measured from the center A upon AD, being $= \delta$; and *y*, at right angles thereto, $= \beta$.

If C be $= 0$ (that is, if *c* be $= b$), *x* will be equal to the invariable quantity *m*; the projected track, a *right line* parallel to AB; and the track upon the surface of the sphere, a *lesser circle* in a plane parallel to the plane of the great circle BC.

If C be $=$ D, y will be equal to the invariable quantity n; the projected track, a *right line* parallel to AD; and the track on the surface of the sphere, a *lesser circle* in a plane parallel to the plane of the great circle CD.

If Dm^2 be $= Ca^2$ the projected track will be the *right line* AR, and $y = \overline{\frac{B}{C}}\Big|^{\frac{1}{2}} \times x$; the track upon the surface of the sphere being the *great circle* CR.

In all other cases in this projection, the track will be an *hyperbola* whose center is A, semi-axis $Aa = \overline{\frac{Ca^2 \backsim Dm^2}{B}}\Big|^{\frac{1}{2}}$, and the other semi-axis $= \overline{\frac{Ca^2 \backsim Dm^2}{C}}\Big|^{\frac{1}{2}}$; the right line AR being always an *asypmtote*.

Fig. 4. When the track is projected on a plane ACD, to which the radius AB is perpendicular (the point D being the vertex as before) the equation thereof will be $y^2 = \frac{D}{C} \times \overline{m^2 - x^2}$; x, measured from the center A upon AD, being $= \delta$ (as before); and y, at right angles thereto $= \gamma$. This projection of the track of the pole will therefore always be an *ellipsis* a b (or a *circle*) whose center is A; semi-axis A a $= m$; and the other semi-axis $= \overline{\frac{D}{C}}\Big|^{\frac{1}{2}} \times m$: except c be $= b$; in which case the projected track will be a *right line* a b parallel to AC.

Fig. 5. Moreover, the equation of the track projected on the plane ABC, to which the radius AD is perpendicular, will be $y^2 = \frac{D}{B} \times \overline{n^2 - x^2}$; x, measured from the center A upon AB, being $= \beta$; and y, at right angles thereto, $= \gamma$. The track of the pole in this projection will therefore always be an *ellipsis*

a b (or a *circle*) whose center is A; semi-axis A a $=n$; and the other semi-axis $=\overline{\frac{D}{B}}\Big|^{\frac{1}{2}}\times n$; except c be $=d$; in which case the projected track will be a *right line* a b parallel to AC.

With regard to the permanent axes of rotation of our parallelopipedon, it appears, by my *Mathematical Memoirs*, that if two of its dimensions be equal (that is, when the body is a *square prism*), any line passing through the center of gravity of the body, in a plane to which the other dimension is perpendicular, will be a permanent axis of rotation; as will the line passing through that center, at right angles to that plane. If all the three dimensions be equal (that is, when the body is a *cube*), any line whatever passing through the center of gravity of the body will be a permanent axis of rotation.

It is observable, that the momentum of rotation of the body, about the momentary axis, is found by computation always $=\frac{e}{a^2}\times\overline{b^2m^2+c^2a^2+d^2n^2}$, e denoting the angular velocity. But $\frac{f}{a^2}\times\overline{b^2m^2+c^2a^2+d^2n^2}$ is the initial momentum of rotation. Therefore, considering the momentum of rotation as invariable, the angular velocity will be invariable, e being always $=f$, which here denotes the initial angular velocity.

Our next business is to find the length of the track described by the momentary pole (P), upon the spherical surface; and the velocity of the pole in that track.

Fig. 2, 3. It appearing, that the motive force E is $=\frac{Me^2}{3a^5}\times\overline{Dm^2-Ca^2}\times\frac{\gamma}{g}$, and the motive force $\overset{\shortparallel}{E}=\frac{Me^2}{3a^4g}\times\sqrt{Dm^2-C\gamma^2}\times\sqrt{Dn^2-B\gamma^2}$; we find $F=\sqrt{E^2+\overset{\shortparallel}{E}{}^2}$ (the force compounded of those two forces) $=\frac{Me^2}{3a^5}\times\sqrt{D^2m^2n^2-BCa^2\gamma^2}$; and, F being to E as a to the fine of the angle pPu, it follows, that the fine of pPu will be $= \frac{\overline{Dm^2 - Ca^2} \times a\gamma}{g\sqrt{D^2m^2n^2 - BCa^2\gamma^2}}$, and its cofine $= \frac{a^2\sqrt{Dm^2 - C\gamma^2} \times \sqrt{Dn^2 - B\gamma^2}}{g\sqrt{D^2m^2n^2 - BCa^2\gamma^2}}$. Therefore, that cofine being to radius as $\left(\frac{a\dot{\gamma}}{g}\right)$ the fluxion of the arc PQ to $(\dot{z})$ the fluxion of the polar track on the fpherical furface, $\dot{z}$ will be $= \frac{\dot{\gamma}\sqrt{D^2m^2n^2 - BCa^2\gamma^2}}{\sqrt{Dm^2 - C\gamma^2} \times \sqrt{Dn^2 - B\gamma^2}}$. Now, $PpLN$ being a quadrant of a great circle (touching the faid polar track at P), and NAN a diameter of that circle; if we put w to denote the diftance of any particle (p) of the parallelopipedon from that diameter, and G to denote the accelerative force of any fuch particle when w is $= a$; the motive force F $\left(= \frac{Me^2}{3a^5} \times \sqrt{D^2m^2n^2 - BCa^2\gamma^2}\right)$, computed above, will be $= \frac{G}{a^2} \times$ *the fum of all the* $w^2 \times p$; which fum, by computation, is found $= \frac{M}{3} \times \frac{\overline{d^2 + b^2} \cdot D^2m^2n^2 - \overline{b^2m^2 + c^2a^2 + d^2n^2} \cdot BC\gamma^2}{D^2m^2n^2 - BCa^2\gamma^2}$. Confequently, G will be $= \frac{e^2}{a^3} \times \frac{\overline{D^2m^2n^2 - BCa^2\gamma^2}|^{\frac{3}{2}}}{\overline{d^2 + b^2} \cdot D^2m^2n^2 - \overline{b^2m^2 + c^2a^2 + d^2n^2} \cdot BC\gamma^2}$. But, by what I have done in the *Philofophical Tranfactions* for the year 1777, $\frac{aG}{e}$ will be $= v =$ the velocity wherewith the momentary pole changes its place in the fpherical furface to which it is referred. Therefore,

v will be $= \frac{e}{a^2} \times \frac{\overline{D^2m^2n^2 - BCa^2\gamma^2}|^{\frac{3}{2}}}{\overline{d^2 + b^2} \cdot D^2m^2n^2 - \overline{b^2m^2 + c^2a^2 + d^2n^2} \cdot BC\gamma^2}$; and $\frac{\dot{z}}{v} = \dot{T}$, the fluxion of the time $= \frac{a^2\dot{\gamma}}{e} \times \frac{\overline{d^2 + b^2} \cdot D^2m^2n^2 - \overline{b^2m^2 + c^2a^2 + a^2n^2} \cdot BC\gamma^2}{\sqrt{Dm^2 - C\gamma^2} \cdot \sqrt{Dn^2 - B\gamma^2} \times \overline{D^2m^2n^2 - BCa^2\gamma^2}}$

$$=\frac{b^2m^2+c^2a^2+d^2n^2}{e}\times\left\{\begin{array}{l}\frac{\dot{\gamma}}{\sqrt{Dm^2-C\gamma^2}\times\sqrt{Dn^2-B\gamma^2}}+\\ \frac{\frac{Dm^2-Ca^2}{b^2m^2+c^2a^2+d^2n^2}\times D^2m^2n^2\dot{\gamma}}{\sqrt{Dm^2-C\gamma^2}\times\sqrt{Dn^2-B\gamma^2}\times\overline{D^2m^2n^2-BCa^2\gamma^2}}\end{array}\right.$$: which,

when Dm^2 is $=Ca^2$, becomes $=\frac{\overline{d^2+b^2}\,.\,a^2\dot{\gamma}}{e\sqrt{BC}\,.\,\overline{a^2-\gamma^2}}$.

It is evident, that $\frac{d^2+b^2}{2e\sqrt{BC}}\times a\times$ hyp. log. of $\frac{a+\gamma}{a-\gamma}$, the value of T in that particular case, will be *infinite* when γ is $=a$; and this conclusion agrees with what is said above respecting the motion of the momentary pole along the great circle CR (fig. 2. and 3.).

I have not found, that the value of T will, in general, be assigned by the *arcs of the conic sections*; but my Tables * shew, that it will be so assigned when Dm^2 is $=Ba^2$, and in some other particular cases.

We have still to investigate the track of the momentary pole in the *immoveable* concave spherical surface, which we must conceive to surround our *moveable* convex spherical surface, supposing the center of both those surfaces to coincide with the centers of gravity of our parallelopipedon: which central point is always in this disquisition supposed at rest.

Let AL be the projection of part of a great circle CL, at right angles to the great circle PpLN; then will the sine of the arc CL be $=\frac{\overline{Dm^2-Ca^2}\times\gamma}{\sqrt{D^2m^2n^2-BCa^2\gamma^2}}$; its cosine $=$ $\frac{D^{\frac{1}{2}}\sqrt{Da^2m^2n^2-\overline{Dm^4-Ca^2m^2+Ca^2n^2}\,.\,\gamma^2}}{\sqrt{D^2m^2n^2-BCa^2\gamma^2}}$; and, the fluxion of that sine being $=\frac{\overline{Dm^2-Ca^2}\times D^2m^2n^2\dot{\gamma}}{\overline{D^2m^2n^2-BCa^2\gamma^2}|^{\frac{3}{2}}}$, the fluxion of that arc (CL) will be

* *Mathematical Memoirs*, published in 1780.

$$=\frac{\overline{Dm^2-Ca^2}\times D^{\frac{3}{2}}am^2n^2\dot{\gamma}}{\overline{D^2m^2n^2-BCa^2\gamma^2}\times\sqrt{Da^2m^2n^2-\overline{Dm^4-Ca^2m^2+Ca^2n^2}\cdot\gamma^2}}.$$ Consequently, the fine of the arc PL being $=\frac{a^2\sqrt{\overline{Dm^2-C\gamma^2}}\times\sqrt{\overline{Dn^2-B\gamma^2}}}{D^{\frac{1}{2}}\sqrt{Da^2m^2n^2-\overline{Dm^4-Ca^2m^2+Ca^2n^2}\cdot\gamma^2}}$; and this fine being to radius as the fluxion of the arc CL to the meafure of the angle of contact of the polar track on the *moveable* fpherical furface with a great circle, we find that meafure

$$=\frac{\overline{Dm^2-Ca^2}\times D^2m^2n^2\dot{\gamma}}{\sqrt{\overline{Dm^2-C\gamma^2}}\times\sqrt{\overline{Dn^2-B\gamma^2}}\times\overline{D^2m^2n^2-BCa^2\gamma^2}}=\frac{\overline{Dm^2-Ca^2}\times D^2m^2n^2\dot{z}}{\overline{D^2m^2n^2-BCa^2\gamma^2}|^{\frac{3}{2}}}.$$

The meafure of the angle of contact of the track of the momentary pole, in the *immoveable* fpherical furface, with a great circle, will accordingly be

$$\dot{e}T-\frac{\overline{Dm^2-Ca^2}\times D^2m^2n^2\dot{z}}{\overline{D^2m^2n^2-BCa^2\gamma^2}|^{\frac{3}{2}}}=\frac{\overline{b^2m^2+c^2a^2+d^2n^2}\cdot\dot{z}}{\sqrt{D^2m^2n^2-BCa^2\gamma^2}}:$$ by means of which meafure we may defcribe, by points, the track of the momentary pole in the fpherical furface laft mentioned.

There are other methods of finding that track; but I know none that is lefs difficult than this method, or in any refpect more fatisfactory.

The radius of the leffer circle, which is the circle of curvature of the polar track in our immoveable fpherical furface, will be =

$$\frac{a\dot{z}}{\sqrt{\dot{z}^2+\text{fq. of the meaf. of the ang. of cont.}}}=\frac{\sqrt{D^2m^2n^2-BCa^2\gamma^2}}{\sqrt{\overline{b^2+c^2}|^2\cdot m^2+\overline{c^2+d^2}|^2\cdot n^2-BC\gamma^2}}.$$

When B is = o, or very fmall in comparifon with D, and Dm^2 is lefs than Ca^2, the laft mentioned radius will be equal, or nearly equal, to the invariable quantity $\frac{Dmn}{\sqrt{\overline{b^2+d^2}|^2\cdot m^2+4d^4n^2}}$; the track of the pole in the immoveable fpherical furface being then exactly, or very nearly, a leffer circle. At the fame time, the

the polar track upon the moveable spherical surface will be exactly, or very nearly, a lesser circle whose radius is m.

When C is $=0$, or very small in comparison with D, and Dm^2 is greater than Ca^2, the track of the pole in the immoveable spherical surface will be exactly, or very nearly, a lesser circle whose radius is $= \frac{Dmn}{\sqrt{4c^4m^2 + \overline{c^2+a^2}|^2 \cdot n^2}}$; and then the polar track upon the moveable spherical surface will be exactly, or very nearly, a lesser circle whose radius is n.

Whatever the curves may be which the momentary pole shall describe in those two spherical surfaces, the track upon the moveable surface will always touch and roll along the track in the immoveable surface (whilst the common center of both surfaces remains at rest), in the manner described in my Paper in the *Philosophical Transactions* for the year 1777; the velocity of the point of contact being equal to the value of v computed above, which velocity when B is $=0$, or very small in comparison with D, and Dm^2 is less than Ca^2, will be exactly, or very nearly, $= \frac{d^2-b^2}{d^2+b^2} \times \frac{mne}{a^2}$; and when C is $=0$, or very small in comparison with D, and Dm^2 is greater than Ca^2, that velocity will be exactly, or very nearly, $= \frac{d^2-c^2}{d^2+c^2} \times \frac{mne}{a^2}$.

The polar track upon the moveable spherical surface will always roll along the convexity of the track in the immoveable spherical surface; the convexity or concavity of the former being turned towards the convexity of the latter, according as Dm^2 is greater or less than Ca^2. Which track in the immoveable spherical surface, when it is not circular, will touch a certain circle as often as γ, during the motion, shall become $=0$; and likewise another parallel circle

circle as often as γ ſhall become equal to $\overline{\frac{D}{C}}\big|^{\frac{1}{2}} \times m$, or $\overline{\frac{D}{B}}\big|^{\frac{1}{2}} \times n$; the parts of the track between the points of contact being perfectly ſimilar. If Dm^2 be $= Ca^2$ (Dn^2 being then $= Ba^2$, and conſequently $\overline{\frac{D}{C}}\big|^{\frac{1}{2}} \times m = \overline{\frac{D}{B}}\big|^{\frac{1}{2}} \times n = a$), the ſaid track will make an infinite number of revolutions about a certain point, continually approaching nearer and nearer thereto, without arriving thereat in any finite time, though the length of the ſpiral ſo deſcribed cannot exceed a certain finite qnantity.

M. Euler has computed, that if the motive forces to turn the revolving body about AB, AC, AD, be reſpectively denoted by H, I, K;

$$\text{H will be} = \frac{M}{3} \cdot \frac{d^2+c^2}{a^3\dot{T}} \times \text{flux. of } e\beta - \frac{M}{3a^5} \cdot Be^2\gamma\delta,$$

$$\text{I} \quad = \frac{M}{3} \cdot \frac{d^2+b^2}{a^3\dot{T}} \times \text{flux. of } e\gamma + \frac{M}{3a^5} \cdot De^2\beta\delta,$$

$$\text{K} \quad = \frac{M}{3} \cdot \frac{c^2+b^2}{a^3\dot{T}} \times \text{flux. of } e\delta - \frac{M}{3a^5} \cdot Ce^2\beta\gamma;$$

γ being ſuppoſed to decreaſe as T increaſes: and he has put the value of each of thoſe forces (H, I, K) = o. In doing ſo, it ſeems to me, that he has erroneouſly aſſumed equations as generally true, which are only ſo in a particular caſe. For $\frac{M}{3a^5} \cdot Be^2\gamma\delta$ is the motive force to turn the body about AB, ariſing from the centrifugal force of its particles revolving about the momentary axis AP, ſuppoſing the pole to keep its place; and $\frac{M}{3} \cdot \frac{d^2+c^2}{a^3\dot{T}} \times$ flux. of $e\beta$ is the value of the motive force requiſite to cauſe the *whole* variation of the velocity $\left(\frac{e\beta}{a}\right)$ about AB. But the firſt mentioned force *alone* does not, in general,

cauſe

cause *all* the variation of the velocity about AB; that velocity varies in consequence of the evagation of the pole P; and that evagation is caused by the motive forces urging the body to turn about AB, AC, AD, *conjunctly*. Therefore the motive force $\frac{M}{3a^5} \cdot Be^2\gamma\delta$ about AB *only* will not, in general, be equal to $\frac{M}{3} \cdot \frac{d^2+c^2}{a^3\dot{T}} \times$ flux. of $e\beta$, the value of the *whole* motive force requisite to cause the variation of the velocity $\frac{e\beta}{a}$, as M. EULER reckoned.

The like objection may, I conceive, be justly made to his other two equations similar to that which is here particularly adverted to.

M. D'ALEMBERT's radical errors, in treating this subject, appear to me nearly similar to M. EULER's.

Other arguments may be adduced to prove, that the equations assumed by those gentlemen are not well founded. If the forces to turn the body about the lines AB, AC, AD were each = o, the velocities about those lines must each remain invariable; but it seems absolutely impossible that they can ever remain so, whilst the angles which those lines make with the momentary axis are each continually varying. Moreover, according to their conclusions, the tangent at P to the track of polar evagation, upon the moveable spherical surface, will not always be perpendicular to the direction in which the pole P will be urged to turn by the joint centrifugal force of the particles of the revolving body; whereas it is proved, I presume, beyond a doubt, in my Paper above-mentioned, that the said track will always be intersected at right angles by the direction in which the momentary pole shall, at any instant of time, be urged to turn by the force causing its evagation.

If we resolve each of the three forces H, I, K, into two others; the one to turn the body about the diameter NAN, and the other to turn it about the momentary axis PAP, at right angles to that diameter; the forces to turn it in the last mentioned direction, arising from the said forces H, I, K, will be

$$\frac{\beta H}{a} = \frac{M}{3} \cdot \frac{d^2+c^2}{a^4\dot{T}} \times \beta \text{ flux. of } e\beta - \frac{M}{3a^6} \cdot Be^2\beta\gamma\delta,$$

$$\frac{\gamma I}{a} = \frac{M}{3} \cdot \frac{d^2+b^2}{a^4\dot{T}} \times \gamma \text{ flux. of } e\gamma + \frac{M}{3a^6} \cdot De^2\beta\gamma\delta,$$

$$\frac{\delta K}{a} = \frac{M}{3} \cdot \frac{c^2+b^2}{a^4\dot{T}} \times \delta \text{ flux. of } e\delta - \frac{M}{3a^6} \cdot Ce^2\beta\gamma\delta.$$

The sum of these forces, it is obvious, must be $=0$; the direction wherein they are supposed to act being at right angles to that in which the body will be actually urged to turn by the joint centrifugal force of its particles, and that being the only force whereby the motion of the body is supposed to be affected: which sum $(B+C-D \text{ being} = 0)$ is, when divided by $\frac{M}{3a^4\dot{T}}$,

$$=\left\{\begin{matrix} \overline{d^2+c^2} \cdot \beta^2\dot{e} + \overline{d^2+b^2} \cdot \gamma^2\dot{e} + \overline{c^2+b^2} \cdot \delta^2\dot{e} \\ \overline{d^2+c^2} \cdot e\beta\dot{\beta} + \overline{d^2+b^2} \cdot e\gamma\dot{\gamma} + \overline{c^2+b^2} \cdot e\delta\dot{\delta} \end{matrix}\right\} = 0.$$

But $\beta\dot{\beta}+\gamma\dot{\gamma}+\delta\dot{\delta}$ being before found $=0$, we have $\overline{d^2+c^2+b^2} \times \overline{\beta\dot{\beta}+\gamma\dot{\gamma}+\delta\dot{\delta}} = 0$; and $b^2\beta\dot{\beta}+c^2\gamma\dot{\gamma}+d^2\delta\dot{\delta}$ being also found $=0$; it evidently follows, that

$$\overline{d^2+c^2} \cdot \beta\dot{\beta} + \overline{d^2+b^2} \cdot \gamma\dot{\gamma} + \overline{c^2+b^2} \cdot \delta\dot{\delta} \text{ will be} = 0.$$

Therefore $\overline{d^2+c^2} \cdot \beta^2\dot{e} + \overline{d^2+b^2} \cdot \gamma^2\dot{e} + \overline{c^2+b^2} \cdot \delta^2\dot{e}$ will be $=0$: consequently $\dot{e}$ will be $=0$, and e invariable; which agrees with what is said above respecting the momentum of rotation.

The other forces arising by resolution from the forces H, I, K, to turn the body about the diameter NAN, will be

$$-\frac{B\gamma\delta H}{S} = -\frac{M}{3a^3}\cdot\frac{d^2+c^2}{S\dot{T}}\cdot Be\gamma\delta\dot{\beta}+\frac{Me^2}{3a^5S}\cdot B^2\gamma^2\delta^2,$$

$$\frac{D\beta\delta I}{S} = \frac{M}{3a^3}\cdot\frac{d^2+b^2}{S\dot{T}}\cdot De\beta\delta\dot{\gamma}+\frac{Me^2}{3a^5S}\cdot D^2\beta^2\delta^2,$$

$$-\frac{C\beta\gamma K}{S} = -\frac{M}{3a^3}\cdot\frac{c^2+b^2}{S\dot{T}}\cdot Ce\beta\gamma\dot{\delta}+\frac{M}{3a^5S}\cdot C^2\beta^2\gamma^2;$$

$$S \text{ being} = \sqrt{D^2m^2n^2 - BCa^2\gamma^2}.$$

And, no external force being suppofed to act on the body, it follows, that the fum of thefe three forces muft be $=0$: therefore we may infer, that

$\dot{T}$ will be $=\frac{a^2}{e}\times\frac{\overline{d^4-c^4}\cdot\gamma\delta\dot{\beta}-\overline{d^4-b^4}\cdot\beta\delta\dot{\gamma}+\overline{c^4-b^4}\cdot\beta\gamma\dot{\delta}}{B^2\gamma^2\delta^2+D^2\beta^2\delta^2+C^2\beta^2\gamma^2}$; which agreeing with the value of $\dot{T}$ found above, the truth of our preceding procefs is thus confirmed.

The force $\frac{Me^2}{3a^5S}\times\overline{B^2\gamma^2\delta^2+D^2\beta^2\delta^2+C^2\beta^2\gamma^2}$, arifing from thofe three forces, is the *whole* joint centrifugal force of the particles of the revolving body, to turn it about the diameter NAN the way it will actually be urged to turn by fuch force; the value whereof fo computed will be $\left(=\frac{Me^2}{3a^5}\times\sqrt{D^2m^2n^2-BCa^2\gamma^2}=\frac{Me^2S}{3a^5}\right)$ equal to the value of the force F computed above, both being confidered as urging the body to turn in the fame direction. And the quantity

$$\frac{Me}{3a^3S\dot{T}}\times\overline{\overline{d^2+c^2}\cdot\gamma\delta\dot{\beta}-\overline{d^2+b^2}\cdot D\beta\delta\dot{\gamma}+\overline{c^2+b^2}\cdot C\beta\gamma\dot{\delta}}$$

$\left(=\frac{e\dot{z}}{a^3\dot{T}}\times\right.$ *the fum of all the* $w^2\times p$) is the value of the motive force which, acting in that very direction, is requifite to caufe the momentary pole to change its place as above defcribed. Thus we fee diftinctly how the equation arifes, by which the value of $\dot{T}$ is juft now determined.

I do

I do not find that the refolving the forces H, I, K, in any other manner will conduce to the attainment of any ufeful conclufion.

It appears, by what is done above, that the force

$$\text{H is} = \frac{\text{M}e}{3a^3\text{R}\dot{\text{T}}} \times \text{CD}^2 m^2 \beta^2 \dot{\beta},$$

$$\text{I is} = \frac{\text{M}e}{3a^3\text{R}\dot{\text{T}}} \times \text{BC} \cdot \overline{\text{C}n^2 - \text{B}m^2} \cdot \gamma^2 \dot{\gamma},$$

$$\text{K} = \frac{-\text{M}e}{3a^3\text{R}\dot{\text{T}}} \times \text{BD}^2 n^2 \delta^2 \dot{\delta};$$

$$\text{R being} = \text{B}^2\gamma^2\delta^2 + \text{D}^2\beta^2\delta^2 + \text{C}^2\beta^2\gamma^2.$$

And it is obvious, that each of the three laft mentioned forces will be = o, if any two of the quantities b, c, d, be equal; two of the values of thofe forces then vanifhing, by reafon of that equality; and the third value alfo vanifhing by either $\dot{\beta}$, $\dot{\gamma}$, or $\dot{\delta}$, being at the fame time = o. Therefore, in that cafe it happens, that M. EULER's computation agrees with mine: in every other cafe, I am clearly of opinion, his conclufions are not true. The fame may be faid of M. D'ALEMBERT's conclufions refpecting the fame propofition.

The evagation of the pole of a revolving body confidered above, does not arife from gravity, the attraction of any other body, or any external impulfe whatever; but is only the confequence of the *inertia of matter*, and muft neceffarily enfue, according to the theory here explained, in every body in the univerfe, after having been made to revolve, without reftraint, about any line paffing through its center of gravity, that is not *a permanent* axis of rotation.

The Earth being neither uniformly denfe nor a perfect fpheroid muft, in ftrictnefs, be confidered as having only three permanent axes of rotation, agreeably to what I have proved in my *Mathematical Memoirs*; and, as it is difturbed in its rotatory motion by the attraction of the fun and moon (and

other

other bodies in our system); it follows; that it will not continually revolve about either of those axes, but will revolve, or endeavour to revolve, about successive momentary axes, as shewn above. If then its three permanent axes of rotation be called its *first*, *second*, and *third* axes; and the poles of its *first* axis be those about which its momentary poles are carried according to our theory; the *second* and *third* axes will be in the plane of its equator, the three being at right angles to each other. Therefore, with respect to the above theory, this terrestrial mass must be considered of such a form, that its equator, and any section parallel thereto, shall rather be elliptical than circular. And, denoting its first, second, and third axes by b, c, d, respectively, observations evince, that the difference $c-b$ will be much greater than the difference $d-c$. Whence it follows, that (supposing the earth's rotatory motion to be disturbed *only* by the centrifugal force arising from the *inertia* of its own particles) the track of polar evagation with us will be nearly circular, and the radius of the limiting circle very small, whether we have regard to the moveable or immoveable spherical surface referred to above; but that, in the latter surface, such circle will be much less than in the former: and it moreover follows, that the concavity of the track upon the moveable surface will continually touch and roll along the convexity of the track in the immoveable surface.

In other planets, the tracks of polar evagation may, from a similar cause, be very different. The theory above explained evidently proves, that their axes of rotation may possibly vary greatly in position, merely through the *inertia of matter*; whilst Providence has so ordered it, that the position of the axes of rotation of this planet shall, by that cause, be but very little altered.

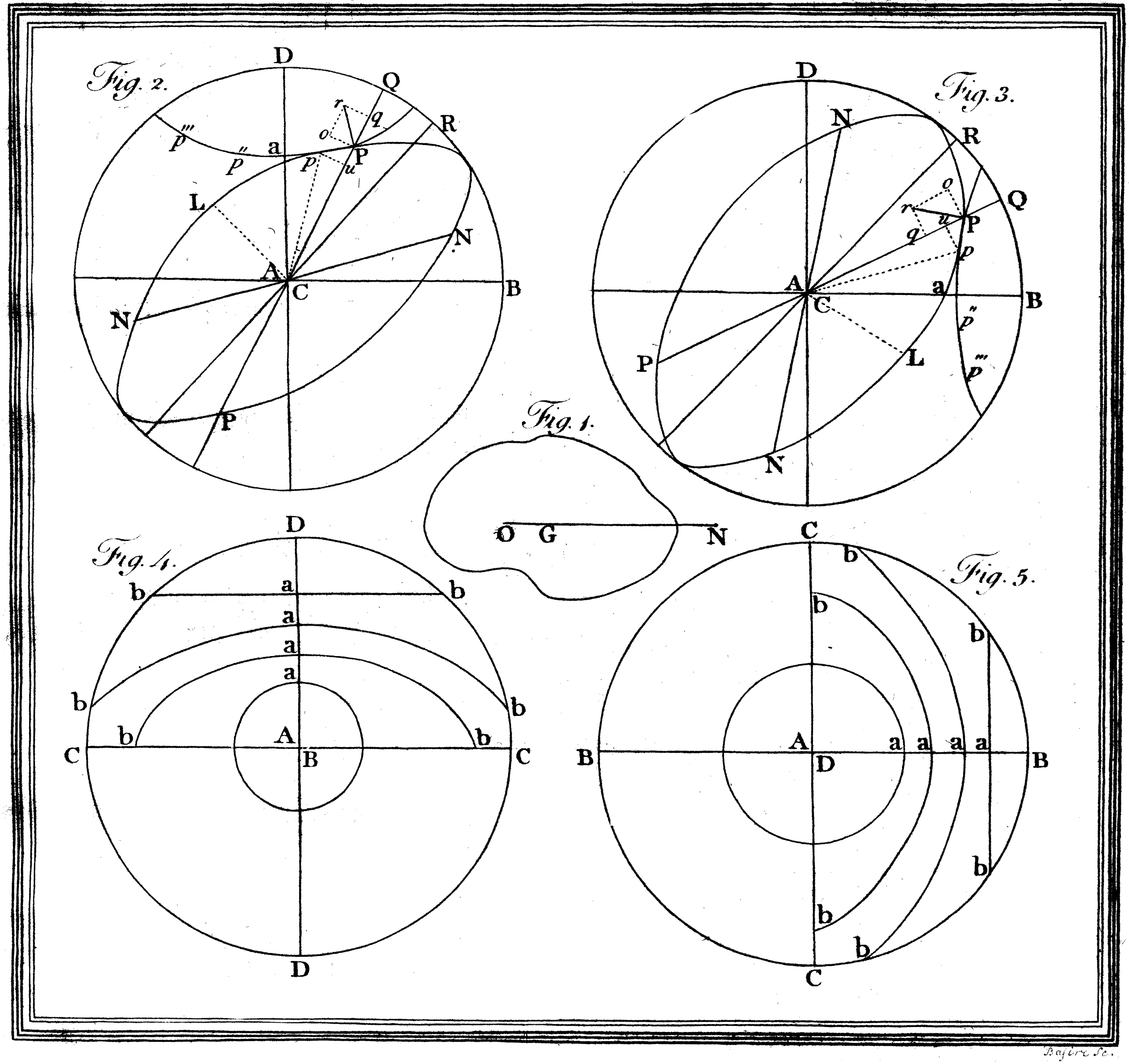
Fig. 2.
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Fig. 3.
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Fig. 1.
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Fig. 4.
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Fig. 5.
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